On Mappings of Constant Multiplicity

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Abstract The main question considered in this paper concerns constructing the mapping of constant odd multiplicity in and onto an open ball of Euclidean space, provided that on the boundary of the ball the mapping is a homeomorphism.

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Definition. *The mapping $f\colon D \longrightarrow D_1$ of the domains is called proper if the preimage of every compact is a compact.*

In [1], it is shown that restriction of a continuous mapping of a closed domain on its interior is proper iff the images of the boundary and the interior of the domain do not intersect.

The aim of this paper is to give a partial answer to the following [2] question:

Question. *Does there exist for every proper mapping $f\colon D \longrightarrow D_1$ (D, D_1 are domains of n-dimensional manifolds) a proper mapping g homotopic to f, such that every point of the image $g(D)$ has no more than $|degfa| + 2$ preimage points (degf is the degree [1] of the mapping f)?*

We study a possibility of constructing a mapping of the constant odd multiplicity on an open ball of Euclidean space $\mathbb{R}^n$, provided that on the boundary of the ball this mapping is a homeomorphism.

Theorem 1. *There exists a continuous mapping of the n-dimensional ball B^n to itself, which is a homeomorphism on the boundary of the ball, and every internal point of the ball has exactly k preimages, where k is an odd positive number.*

Proof. We use a function given on the open interval (0,1), which was built by J. Mioduszewskii [3].

At first, there is a function p_{2r+1}, $r = 1, 2, \ldots$, defined on the closed interval [0,1], such that

$$p_{2r+1}(j/(2r+1)) = \begin{cases} 0 & \text{for} \quad j = 0, 2, \ldots, 2r, \\ 1 & \text{for} \quad j = 1, 3, \ldots, 2r+1, \end{cases}$$

and on the interval $j/(2r+1) < x < (j+1)/(2r+1),\ j = 1, 2, \ldots, 2r$, it is linear joining suitable points 0 and 1.

Further, we shall define a continuous function q_{2r+1} on the closed interval [0,1], such that $q_{2r+1}(0) = 0$,

$$q_{2r+1}(x) = 2^{-m}(1 + p_{2r+1}(2^m(x - 2^{-m}))) \quad \text{for} \quad 2^{-m} \le x \le 2^{-m+1},\ m = 1, 2, \ldots$$

Next we define the function

$$s_{2r+1}(x) = \begin{cases} q_{2r+1}(x+1) - 1 & \text{for} \quad -1 < x \le 0, \\ 1 - q_{2r+1}(1-x) & \text{for} \quad 0 \le x < 1, \quad r = 1, 2, \ldots, \end{cases}$$

determined on the open interval $-1 < x < 1$. Values of the function s_{2r+1} at the points -1 and 1 are defined by the equalities $s_{2r+1}(-1) = -1, s_{2r+1}(1) = 1$. As a result we obtained a continuous function on the closed interval $B^1 = [-1, 1]$ (the one-dimensional ball), which is a one-to-one mapping on the boundary and such that each point of ball interior has exactly the odd number of the preimages $k = 2r + 1 \ge 3$.

To proceed further we consider n-dimensional ball B^n as the suspension SB^{n-1} [4] over the $(n-1)$-dimensional ball B^{n-1} (for instance, two-dimensional ball $B^2 = SB^1 = \{(x, y)||x| + |y| \le 1\}$ is the suspension on the closed interval). As the next step, we spread the function s_{2n+1} on the mapping of the suspension by the formula

$$s_{2r+1}(x,y) = \begin{cases} (1-|y|)s_{2r+1}(x/(1-|y|)) & \text{for} \quad y \neq -1, 1, \\ 0 \quad \text{for} \quad y = -1, 1 \end{cases}$$

The latter is here presented for the mapping of the two-dimensional ball. A similar mapping can be built for every dimension by induction:

$$s_{2r+1}(x_1, x_2, \ldots, x_n) = \begin{cases} (1-|x_n|)s_{2r+1}(x_1/(1-|x_n|), \quad x_2/(1-|x_n|), \ldots, x_{n-1}/(1-|x_n|)), \\ \quad \text{for} \quad x_n \neq -1, 1, \\ 0 \quad \text{for} \quad x_n = -1, 1 \end{cases}$$

It is easy to see that this mapping satisfies the conditions of the theorem for $k \geq 3$, but for $k = 1$ there exists a natural homeomorphism of the ball into itself. □

Corollary 2. *Let a set X be presented as the Cartesian product $X = Y \times B^n$. Then one can define on X a continuous mapping, which is a homeomorphism to $X \times \partial B^n$ and such that each point of $X \times IntB^n$ has exactly k preimages, where k is an odd positive number.*

Remark 3. *From this result, in particular, one follows that for an arbitrary homeomorphism of the Cartesian product $X = Y \times B^n$ onto itself (the multiplicand Y can be an empty set) there exists an internal three-to-one mapping of X onto itself with conservation of the homeomorphism property on the boundary.*

Proposition 4. *A continuous mapping of the closed interval into itself with constant multiplicity more than one does not exist.*

Proof. We suppose that such a mapping exists. Let its multiplicity be $k > 2$. Then the minimal value function is achieved at internal point of the closed interval at least once. Making use of the Darboux theorem about intermediate value it is easy to see that a value close to the minimum occurs at least once over. But, if multiplicity is twofold, then from the previous result one follows that values of this function at endpoints of the closed interval must coincide, and can be taken nowhere at interior points. Without loss of generality, we can expect that at the endpoints of the closed interval this function realizes its minimal values. Then there exists a unique point carrying the maximum of the function, and belonging to the interior of the closed interval. So, based once more on the Darboux theorems, one obtains that the maximal value can occur only once. The obtained above contradiction proves the proposition. □

Remark 5. *If there is no restriction on the mapping of the closed interval to the closed interval, such a mapping exists [3] if $k \neq 2$.*

A mapping of a domain is called an interior mapping if the image of every open set is open, and the set of preimages of an arbitrary point consists of isolated points.

The following theorem concerning estimation of the set of multiple points was received in [5, 6].

Theorem 6. *Every proper mapping of a domain of an n-dimensional manifold onto a domain of another n-dimensional manifold of degree is either an interior mapping or there exists a point from the image possessing not less than $|k| + 2$ true preimages. If the restriction of this function to the interior of the domain is a zero-dimensional mapping, then the set of points of the image, possessing not less than |k|+2 true preimages, contains a subset of complete dimension n.*

Open questions.

1. Does there exist a mapping of the closed ball into itself of a constant multiplicity for $k \geq 3$? It is known that for $k = 2$ such a mapping does not exist [7].

2. Does there exist a mapping of the n-dimensional real projective space in an n-dimensional sphere such that every point of the image has not more than two preimage points for $n \geq 4$? It is known that for $n = 2$ and 3 such a mappings exists [8].

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